SCOTCH
COLLEGE

Scotch College
Semester One Examination, 2010

Question/Answer Booklet

MATHEMATICS
3C/3D Specialist

Section One:
Calculator free

Teacher: ____**Mr Hill**
____**Mr Robb**

Name: SOLUTIONS

Time allowed for this section

Reading time before commencing work: 5 minutes
Working time for this section: 50 minutes

Material required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, eraser, correction fluid, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available
Section One: Calculator-free	6	6	50	40
Section Two: Calculator-assumed	12	12	100	80
				120

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2010*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

3. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil** except in diagrams.

Section One: Calculator-free (40 Marks)

This section has **six (6)** questions. Answer **all** questions. Write your answers in the space provided.
Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

Suggested working time for this section is 50 minutes.

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1. [10 marks]

Given that $z_1 = 3-4i$ and $z_2 = -2-2i$, determine exactly:

(a) $\operatorname{Im}(z_1 + z_2)$ **[1]**

-6 ✓

$1-6i$

(b) $\overline{z_1 z_2}$ **[2]**

$(3+4i)(-2+2i)$
$= -6+6i-8i-8$ ✓
$= -14-2i$ ✓

(c) $\left|\frac{z_1}{z_2}\right|$ **[3]**

$= \left|\frac{3-4i}{-2-2i} \times \frac{-2+2i}{-2+2i}\right| = \left|\frac{-6+6i+8i+8}{4+4}\right|$ ✓

$= \left|\frac{14i+2}{8}\right|$ ✓

$= \frac{1}{4}|7i+1|$

$= \frac{1}{4}\sqrt{49+1} = \frac{1}{4}\sqrt{50} = \frac{5}{4}\sqrt{2}$ ✓

(d) the complex number k such that $\operatorname{Im}(k) = 2\operatorname{Re}(k) - 1$ and $k = \bar{k} + \operatorname{Re}(k) \times i$. **[4]**

let $k = x+yi$
$\operatorname{Im}(k) = 2\operatorname{Re}(k) - 1$
$y = 2x-1$ ① ✓
$k = \bar{k} + \operatorname{Re}(k) \times i$
$x+yi = x-yi+xi$
$2yi = xi$
$2y = x$ ② ✓

subst ② into ①
$y = 2(2y)-1$
$y = 4y-1$
$y = \frac{1}{3}$ ✓ $x = \frac{2}{3}$

$k = \frac{2}{3} + \frac{1}{3}i$ ✓

$\frac{5}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$\frac{5\sqrt{2}}{4}$

2. **[8 marks]**

Find $\frac{dy}{dx}$ for each of the following, simplifying answers wherever possible

(a) $y = \sqrt{\sin 2x}$ **[2]**

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\sin 2x}} \cdot 2\cos 2x$$

$$= \frac{\cos 2x}{\sqrt{\sin 2x}}$$

(b) $y = \tan^2(5 - \pi x)$ **[3]**

$$\frac{dy}{dx} = 2\tan(5-\pi x) \cdot \frac{1}{\cos^2(5-\pi x)} \cdot -\pi$$

(c) $x\cos y + (y+1)^3 = \frac{\pi}{3}$ **[3]**

$$1.\cos y + x(-\sin y)\frac{dy}{dx} + 3(y+1)^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(3(y+1)^2 - x\sin y\right) = -\cos y$$

$$\frac{dy}{dx} = \frac{-\cos y}{3(y+1)^2 - x\sin y}$$

$$= \frac{\cos y}{x\sin y - 3(y+1)^2}$$

4. **[4 marks]**

Find the following indefinite integrals.

(a) $\int x^{\frac{1}{3}} - x^{-\frac{1}{3}} \quad dx$ [2]

$$= \frac{3x^{\frac{4}{3}}}{4} - \frac{3x^{\frac{2}{3}}}{2} + c$$

(b) $\int 4\sin(\cos x).\sin x \quad dx$ [2]

$$= 4\cos(\cos x) + c$$

3. **[5 marks]**

Matrices **A** , **B** and **C** are all 2x2 matrices and $\mathbf{C} = \mathbf{A} - \mathbf{CB}$

Determine **C** given that $\mathbf{A} = \begin{bmatrix} -1 & 6 \\ 11 & 4 \end{bmatrix}$ *and* $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ -5 & 1 \end{bmatrix}$

$$C = A - CB$$

$$C + CB = A$$

$$C(I+B) = A \quad \checkmark$$

$$C\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -5 & 1 \end{bmatrix}\right) = \begin{bmatrix} -1 & 6 \\ 11 & 4 \end{bmatrix}$$

$$C\begin{bmatrix} 2 & 2 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 6 \\ 11 & 4 \end{bmatrix} \quad \checkmark$$

$$C = \begin{bmatrix} -1 & 6 \\ 11 & 4 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -5 & 2 \end{bmatrix}^{-1} \quad \checkmark$$

$$\begin{bmatrix} 2 & 2 \\ -5 & 2 \end{bmatrix}^{-1} = \frac{1}{4+10}\begin{bmatrix} 2 & -2 \\ 5 & 2 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 2 & -2 \\ 5 & 2 \end{bmatrix} \quad \checkmark$$

$$C = \frac{1}{14}\begin{bmatrix} -1 & 6 \\ 11 & 4 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ 5 & 2 \end{bmatrix}$$

$$C = \frac{1}{14}\begin{bmatrix} 28 & 14 \\ 42 & -14 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \quad \checkmark$$

5. [5 marks]

Use proof by induction to prove that $4^n + 2$ is divisible by 3 for $n \in \mathbb{Z}^+$.

R.T.P. = $\frac{4^n+2}{3} = A$, $A \in \mathbb{Z}$

$4^n + 2 = 3A$ ✓

Proof = By mathematical Induction

① Let $n=1$ $LHS = 4^1 + 2$

$= 6$

$= 3(2)$

$= 3A$

$= RHS$ ✓

holds for $n=1$

② If we assume statement holds for $n=k$, then we need to show statement holds for $n=k+1$ ✓

$4^n + 2 = 3B$ $B \in \mathbb{Z}$

$4^{k+1} + 2 = 3B$

$4^k . 4 + 2 = 3B$

$4(3A-2) + 2 = 3B$ ✓

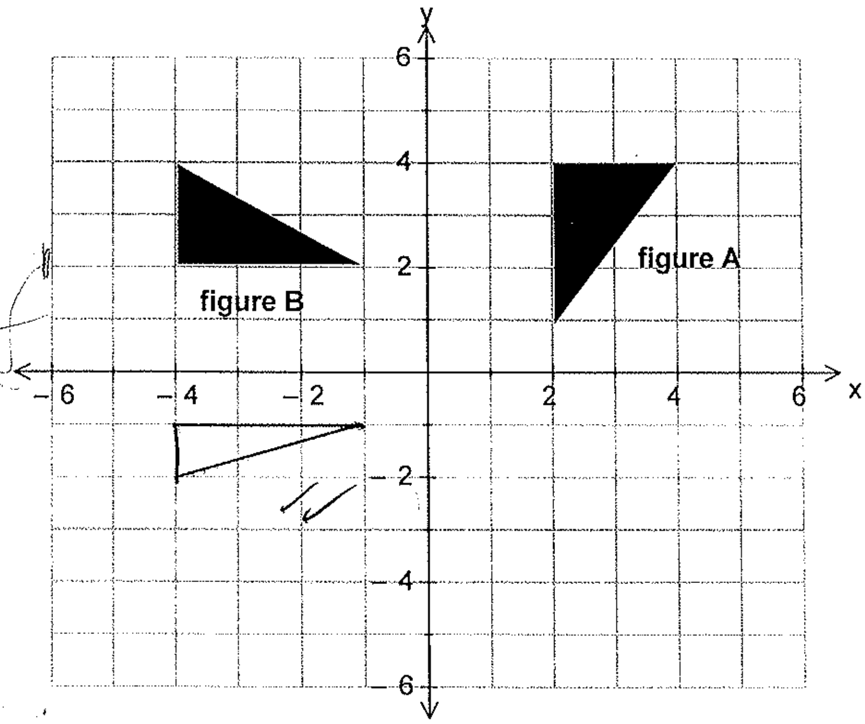
$12A - 8 + 2 = 3B$

$12A - 6 = 3B$

$3(4A-2) = 3B$

Holds for $n=k+1$ where $B = 4A-2$ ✓, $A \in \mathbb{Z}$; $B \in \mathbb{Z}$

6. **[8 marks]**



(a) Write down the 2x2 transformation that would map the points from figure A onto the points from figure B. **[1]**

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ Rotation 90° about Origin.

(b) Find a 2x2 matrix, T, that would firstly *reflect* figure B about the X axis and then transform this image onto a triangle with half the area of figure B. Draw the image on the grid above and label it figure C. **[5]**

-2 wrong

$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$ | $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix}$ | $\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$

Dilation Reflect X-axis OR Dilation Reflection x-axis OR

Could be any of these answers ...

(c) Find a single transformation matrix that will map figure A onto figure C. **[2]**

$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -\frac{1}{2} & 0 \end{bmatrix}$

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MATHEMATICS
3C/3D Specialist

Section Two:
Calculator-assumed

Teacher: ____**Mr Hill**
____**Mr Robb**

Name: SOLUTIONS

Time allowed for this section
Reading time before commencing work: 10 minutes
Working time for this section: 100 minutes

Material required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the Curriculum Council for this examination

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Section Two: Calculator-assumed **(80 Marks)**

This section has **twelve (12)** questions. Answer **all** questions. Write your answers in the space provided.

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- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
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Suggested working time for this section is 100 minutes.

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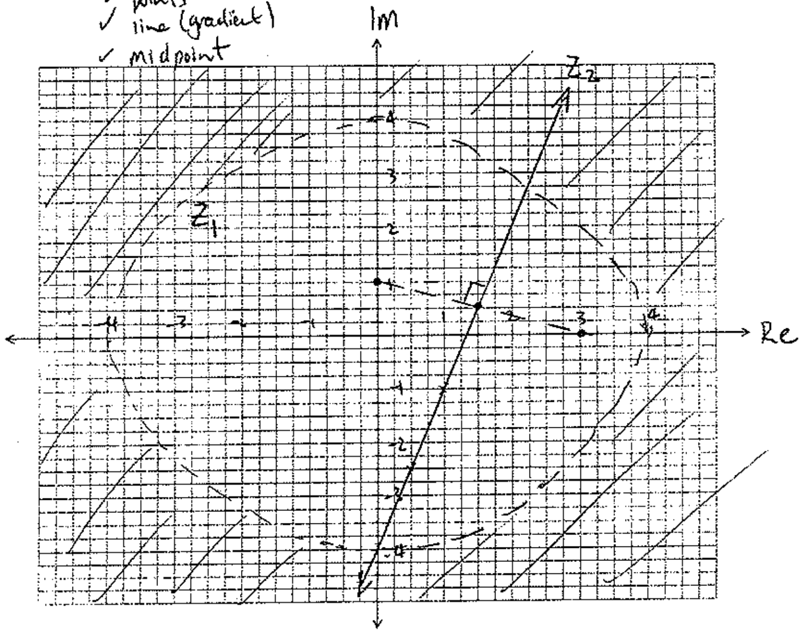
1. **[5 marks]**

Display the following on the single Argand Diagram below.

(a) $|z_1| > 4$ ✓ dotted ✓ shading [2]

(b) $z_2 : |z - i| = |3 - z|$ [3]

✓ points
✓ line (gradient)
✓ midpoint



2. **[4 marks]**

Consider the matrix $P = \begin{bmatrix} n-1 & 4 \\ 3 & n-2 \end{bmatrix}$

(a) Find the det(P) and hence find values of n such that det(P) = 0 [3]

$\text{Det} = ad - bc = (n-1)(n-2) - 4.3$ ✓

$= n^2 - 3n + 2 - 12$

$= n^2 - 3n - 10$ ✓

$0 = n^2 - 3n - 10$

$0 = (n+2)(n-5)$ ✓

$n = -2, 5$

(b) State the conditions that would allow P^{-1} to exist. [1]

P^{-1} exists if $n \neq -2, 5$ ✓

3. **[5 marks]**

Point A has polar co-ordinates $\left(\sqrt{2}, \frac{\pi}{3}\right)$ and point B $\left(3, \frac{4\pi}{5}\right)$

(a) Determine the distance between A and B. [2]

$AB = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)}$

$= \sqrt{(\sqrt{2})^2 + (3)^2 - 2(\sqrt{2})(3)\cos(\frac{\pi}{3} - \frac{4\pi}{5})}$

$= \sqrt{10.113} = 3.18$ to 2d.p. ✓✓

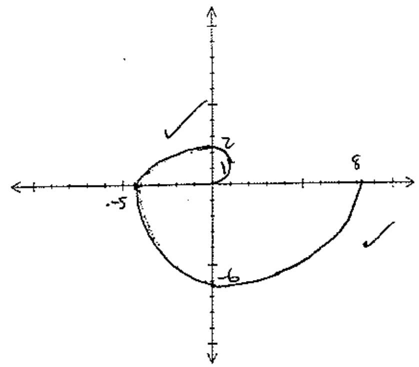
(b) If the point A lies on the line $r = k\theta$, determine k and hence sketch the graph for $0 \leq \theta \leq 2\pi$. [3]

$r = k\theta$

$\sqrt{2} = k(\frac{\pi}{3})$

$k = \frac{3\sqrt{2}}{\pi}$ ✓

$r = \frac{3\sqrt{2}}{\pi}\theta$



4. [7 marks]

Using the substitution provided, solve the following integral. Show full working.

$$\int_0^{\frac{1}{6}} \sqrt{1-9t^2}\, dt \qquad \text{let } t = \frac{\sin\theta}{3}$$

$t = \frac{1}{6}$ $\quad \frac{1}{6} = \frac{\sin\theta}{3}$ $\quad t = 0 \quad \sin\theta = 0$

$\frac{1}{2} = \sin\theta \qquad \underline{\theta = 0}$

$\underline{\theta = \frac{\pi}{6}}$ ✓

$t = \frac{\sin\theta}{3}$

$\frac{dt}{d\theta} = \frac{1}{3}\cos\theta$ ✓

$$\int_0^{\frac{1}{6}} \sqrt{1-9t^2}\, dt$$

$= \int_0^{\frac{\pi}{6}} \sqrt{1-9\left(\frac{\sin\theta}{3}\right)^2} \cdot \frac{1}{3}\cos\theta\, d\theta$ ✓

$= \int_0^{\frac{\pi}{6}} \sqrt{\cos^2\theta}\ \frac{1}{3}\cos\theta\ d\theta$

$= \int_0^{\frac{\pi}{6}} \frac{1}{3}\cos^2\theta\ d\theta$ ✓

$= \frac{1}{3}\int_0^{\frac{\pi}{6}} \frac{1-\cos 2\theta}{2}\ d\theta$ ✓

$= \frac{1}{6}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{6}}$ ✓

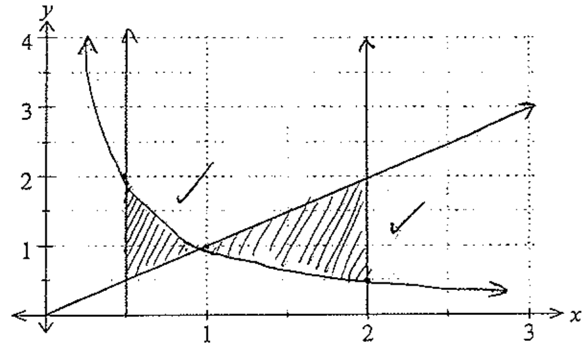
$= \frac{1}{6}\left[\frac{\pi}{6} + \frac{\frac{\sqrt{3}}{2}}{2}\right] - [0]$

$= \frac{\pi}{36} + \frac{\sqrt{3}}{24}$

$= 0.159$ ✓

5. [6 marks]

(a) On the following axes neatly sketch, for $x > 0$, the graphs $y = x$, $y = \frac{1}{x}$ and the lines $x = \frac{1}{2}$ and $x = 2$. Hence, shade the region(s) trapped between the graphs of $y = x$, $y = \frac{1}{x}$ and the lines $x = \frac{1}{2}$ and $x = 2$. [2]



(b) Complete the following expressions for the total area of the region shaded. No integrals may be added to the expression and all integrals shown are to be used. [3]

(i) $\int_{\frac{1}{2}}^{1} \frac{1}{x} - x\ dx + \int_{1}^{2} x - \frac{1}{x}\ dx$

(ii) $\int_{\frac{1}{2}}^{2} \left| \frac{1}{x} - x \right| dx$

(iii) $1\frac{1}{2} \times 1\frac{1}{2} - \int_{\frac{1}{2}}^{2} \left| \frac{1}{y} - y \right| dy$

(c) Calculate the total area of the region shaded. [1]

1.125 unit^2 ✓

6. [6 marks]

(a) Find the vector equation of a line, $\underset{\sim}{r}_2$, passing through the point with position vector $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and perpendicular to the line $\underset{\sim}{r}_1 = \begin{pmatrix} -4 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$ [2]

$$\underset{\sim}{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \checkmark \checkmark$$

(b) Find any position vector(s) that lie on $\underset{\sim}{r}_1$ and have a length of magnitude 6. [4]

$$\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4-\lambda \\ 4 \\ 1+3\lambda \end{pmatrix}$$

$x = -4-\lambda$

$y = 4$

$z = 1+3\lambda$ ✓

$\sqrt{x^2+y^2+z^2} = 6$ ✓

$(-4-\lambda)^2 + (4)^2 + (1+3\lambda)^2 = 36$

$16+8\lambda+\lambda^2+16+9\lambda^2+6\lambda+1 = 36$

$10\lambda^2+14\lambda-3 = 0$

$\lambda = -1.589, 0.1889$ ✓

$\underset{\sim}{r} =$ OR ✓

$$\begin{pmatrix} -2.41 \\ 4 \\ -3.77 \end{pmatrix} \qquad \begin{pmatrix} -4.19 \\ 4 \\ 1.57 \end{pmatrix}$$

7. [6 marks]

The parametric equation of a curve is given as $x = t^4 - t^2$ and $y = t^2 + \ln(t)$. Find the equation of the tangent to this curve when $t = 1$.

$\frac{dx}{dt} = 4t^3 - 2t$ ✓ $\quad \frac{dy}{dt} = 2t + \frac{1}{t}$

$= \frac{2t^2+1}{t}$ ✓

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{2t^2+1}{t} \times \frac{1}{4t^3-2t}$ ✓ $(t=1)$

$= \frac{2(1)^2+1}{1} \times \frac{1}{4(1)^3-2(1)}$

$= \frac{3}{2}$ ✓

$t=1 \quad x=0, y=1$ ✓

$\therefore y = \frac{3}{2}x + 1$ ✓

$2y = 3x + 2$

8. **[10 marks]**

(a) By considering an integer n as either a multiple of 3 or 1 more than a multiple of 3 or 2 more than a multiple of 3;
Use proof by exhaustion to show than if n^2 is a multiple of 3 then n must also be a multiple of 3. [5]

let ① $n = 3k$ or ② $3k+1$ or ③ $3k+2$. $k \in \mathbb{Z}$

then ① $n^2 = (3k)^2$
$= 9k^2$
$= 3(3k^2)$ n & n^2 multiple of 3.

② $n^2 = (3k+1)^2$
$= 9k^2 + 6k + 1$
$= 3(3k^2+2k)+1$ n & n^2 not a multiple of 3

③ $n^2 = (3k+2)^2$
$= 9k^2 + 12k + 4$
$= 3(3k^2+4k+1)+1$ n & n^2 not a multiple of 3.

∴ If n^2 is a multiple of 3, n must be a multiple of 3

(b) Use proof by contradiction to show that the square root of 3 is irrational.
(You may use your result from (a)) [5]

Proof by contradiction:
Let us assume $\sqrt{3}$ is rational.

i.e. $\sqrt{3} = \frac{a}{b}$ where a & $b \in \mathbb{Z}$ s.t. a & b have no common factors

then $3 = \frac{a^2}{b^2}$

$3b^2 = a^2$

If a^2 is a multiple of 3 then from (a) a = multiple of 3.
i.e. $a = 3k$.

∴ $3b^2 = (3k)^2$
$3b^2 = 3(3k^2)$
$b^2 = 3k^2$

If b^2 is a multiple of 3 b is a multiple of 3.
But if a & b have a common factor of 3, we have a contradiction. Contradiction is false hence $\sqrt{3}$ is irrational.

9. **[8 marks]**

The Swanbourne Sweets Store started a promotion to sell three different boxes of a dozen mixed chocolates. Each mixed box contains different quantities of white, dark and milk chocolate. This information is shown in the following table.

	Box 1	Box 2	Box 3
White	8	6	4
Dark	2	2	4
Milk	2	4	4

Swanbourne Sweets purchases white chocolate for \$11 per chocolate, dark chocolate for \$8 per chocolate and milk chocolate for \$9 per chocolate.

Using P to represent the mixed box matrix and Q to represent the cost matrix, use matrix methods to answer the following, clearly indicating the matrix operation used.

(a) Determine the matrix R which represents the cost to Swanbourne Sweets for a mixed dozen of each of the three varieties. [3]

$$R = P \times Q = \begin{bmatrix} 8 & 2 & 2 \\ 6 & 2 & 4 \\ 4 & 4 & 4 \end{bmatrix} \begin{bmatrix} 11 \\ 8 \\ 9 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 122 \\ 118 \\ 112 \end{bmatrix} \end{array} \begin{array}{l} \text{Box 1} \\ \text{Box 2} \\ \text{Box 3} \end{array}$$

(b) One particular consumer orders eight mixed dozens of Box 1, five mixed dozens of Box 2 and six mixed dozens of Box 3. Find a matrix S showing the number of white chocolate, dark chocolate and milk chocolate Swanbourne Sweets requires to fill this order. [2]

$$S = \begin{bmatrix} 8 & 6 & 4 \\ 2 & 2 & 4 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 118 \\ 50 \\ 60 \end{bmatrix} \begin{array}{l} \text{White} \\ \text{Dark} \\ \text{Milk} \end{array}$$

(c) Swanbourne Sweets makes a profit of 24%, 30% and 40% on the sale of each chocolate of white, dark and milk respectively. Determine the sell price matrix T for each type of chocolate and hence the total amount paid by the customer for the order in part (b). [3]

$$T = \begin{bmatrix} 11 \times 1.24 & 8 \times 1.3 & 9 \times 1.4 \end{bmatrix}$$
$$= \begin{bmatrix} 13.64 & 10.4 & 12.6 \end{bmatrix}$$

$$\begin{bmatrix} 13.64 & 10.4 & 12.6 \end{bmatrix} \begin{bmatrix} 118 \\ 50 \\ 60 \end{bmatrix}$$
$$= [\ \$2885.52\ ]$$

10. [7 marks]

(a) Show that the line $x = 2 + t$, $y = -1 + 2t$, $z = 3t$ is parallel to the plane $11x - 4y - z = 0$. [4]

$$\underset{\sim}{r}_1 = \begin{pmatrix} 2+t \\ -1+2t \\ 3t \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \checkmark \qquad \Pi_1 : \underset{\sim}{r} \cdot \begin{pmatrix} 11 \\ -4 \\ -1 \end{pmatrix} = 0 \checkmark$$

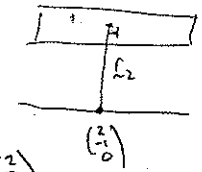
To be parallel the direction vector for r_1 must be perpendicular to the normal vector to the plane Π,

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 11 \\ -4 \\ -1 \end{pmatrix} = 11 + -8 + -3 \checkmark$$
$$= 0$$

$$\therefore \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \perp \begin{pmatrix} 11 \\ -4 \\ -1 \end{pmatrix} \qquad \underset{\sim}{r}_1 \parallel \Pi_1 \checkmark$$

(b) Hence find the line's distance from the plane. [3]

$\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ is a point on $\underset{\sim}{r}_1$



✓

let $\underset{\sim}{r}_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 11 \\ -4 \\ -1 \end{pmatrix}$ be a line perpendicular to plane through $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

$$\underset{\sim}{r}_2 = \begin{pmatrix} 2+11\lambda \\ -1-4\lambda \\ 0-\lambda \end{pmatrix}$$

$$\underset{\sim}{r}_2 \cdot \begin{pmatrix} 11 \\ -4 \\ -1 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2+11\lambda \\ -1-4\lambda \\ -\lambda \end{pmatrix} \cdot \begin{pmatrix} 11 \\ -4 \\ -1 \end{pmatrix} = 0$$

$$22 + 121\lambda + 4 + 16\lambda + \lambda = 0 \checkmark$$
$$138\lambda = -26$$
$$\lambda = \frac{-26}{138} = -\frac{13}{69}$$

$$\frac{13}{69}\left|\begin{pmatrix} 11 \\ -4 \\ -1 \end{pmatrix}\right|$$

$$= 2.21 \text{ units} \checkmark$$

OR.

$\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ a point ✓

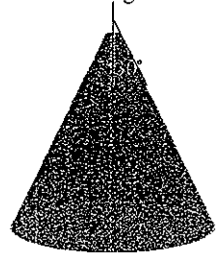
$$d = \frac{|2 \times 11 + -1 \times -4 + 0 \times -1 - 0|}{\sqrt{(11)^2 + (-4)^2 + (-1)^2}} \checkmark$$

$$= \frac{26}{11.747}$$

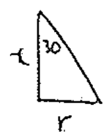
$$= 2.21 \text{ units.} \checkmark$$

11. [6 marks]

A helicopter is descending towards level ground. It has a bright light directed vertically downwards such that the cone of light makes a circle on the ground.



The cone has a semi vertical angle of 30°, as shown. The helicopter descends at a constant rate of 3 m/sec. At what rate is the area of the circle changing when the helicopter is 8 m above the ground?



$$\tan 30 = \frac{x}{r}$$
$$r = \frac{1}{\sqrt{3}} x \checkmark$$
$$\frac{dx}{dt} = -3 \text{ m/s} \checkmark$$

$$A = \pi r^2$$
$$= \pi\left(\frac{x}{\sqrt{3}}\right)^2$$
$$= \pi \frac{x^2}{3} \checkmark$$

$$\frac{dA}{dx} = \frac{2\pi}{3} x \checkmark$$

$$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt} \checkmark$$
$$= \frac{2\pi}{3} x \times -3$$
$$= -\frac{6\pi}{3}(8)$$
$$= -16\pi \text{ m}^2/\text{sec} \checkmark$$

12. **[10 marks]**

(a) If $[g(x)]^2 + 3x = x^2 g(x)$ and $g(2) = 3$, find $g'(2)$. [4]

$2\,g(x)\,g'(x) + 3 = 2x\,g(x) + x^2 g'(x)$

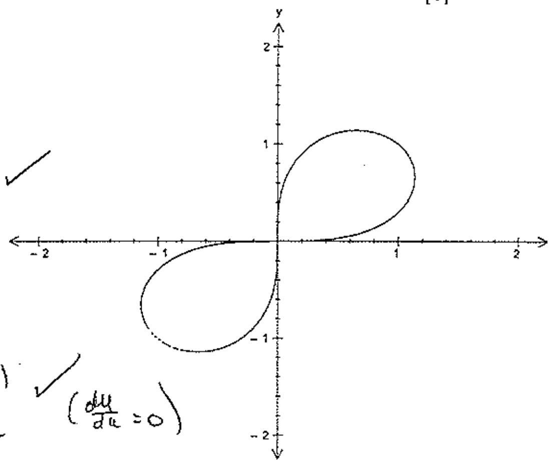
$2g(2)\,g'(2) + 3 = 2(2)\,g(2) + (2)^2 g'(2)$

$2.3.\,g'(2) + 3 = 4 \times 3 + 4\,g'(2)$

$g'(2) = 4.5$

(b) The graph of the equation $(x^2+y^2)^2 = 4xy$ is called a lemniscate and is shown below.

The points on the graph where the tangent is horizontal can be found when $\frac{dy}{dx} = 0$. Use implicit differentiation and algebraic techniques to prove that these points are given by $\pm\left(\frac{3^{\frac{1}{4}}}{2}, \frac{3^{\frac{3}{4}}}{2}\right)$. [6]



$(x^2+y^2)^2 = 4xy$

$2(x^2+y^2)(2x+2y\frac{dy}{dx}) = 4y + 4x\frac{dy}{dx}$

$2x(x^2+y^2) + 2y\frac{dy}{dx}(x^2+y^2) = 2y + 2x\frac{dy}{dx}$

$\frac{dy}{dx}(2y(x^2+y^2) - 2x) = 2y - 2x(x^2+y^2)$

$\frac{dy}{dx} = \frac{2y - 2x(x^2+y^2)}{2y(x^2+y^2) - 2x}$ $\left(\frac{dy}{dx} = 0\right)$

$0 = 2y - 2x(x^2+y^2)$

$y = x(x^2+y^2)$

$y^2 = x^2(x^2+y^2)^2$

$y^2 = x^2(4xy)$

$y = 4x^3$

$(x^2+y^2)^2 = 4xy$

$(x^2+16x^6)^2 = 16x^4$ $(y = 4x^3)$

$x^2 + 16x^6 = 4x^2$

$-3x^2 + 16x^6 = 0$

$x^2(16x^4 - 3) = 0$

$x^2 = 0$ or $x^4 = \frac{3}{16}$

$x = \pm\frac{3^{\frac{1}{4}}}{2}$

$\therefore y = 4\left(\frac{3^{\frac{1}{4}}}{2}\right)^3 = \pm\frac{3^{\frac{3}{4}}}{2}$